Definitions and key facts for section 1.8

A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

The set $\mathbb{R}^n$ is the **domain** of T while $\mathbb{R}^m$ is the **codomain**.

We use the notation

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ and } T : \mathbf{x} \mapsto T(\mathbf{x})$$

to denote that T maps elements of $\mathbb{R}^n$ to elements of $\mathbb{R}^m$, and to denote that the vector $\mathbf{x}$ is mapped to $T(\mathbf{x})$, respectively.

The vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$ (under the transformation T). The set of all images is called the **range** of T.

A transformation T which maps each $\mathbf{x}$ to the product $A\mathbf{x}$ using a fixed matrix A is called a **matrix transformation**.

In general, if A is an $m \times n$ matrix, then $\mathbf{x} \mapsto A\mathbf{x}$ is a matrix transformation with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

A transformation T is **linear** if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T; and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Fact: If T is a linear transformation then

$$T(\mathbf{0}) = \mathbf{0},$$

and

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of T and all scalars c, d. Thus more generally, we have

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p)$$

for any linear combination consisting of vectors from the domain of T.